Going down theorem (All rings are Noetherian)

Note that the characterization of dimension for local rings yields the following easy result about algebras over local rings:

Prop: Let (R,m) be a local ring and S an R-algebra with $mS \neq S$. Then $\operatorname{codim} mS \leq \operatorname{codim} m$.

Pf: If $x_1, \dots, x_d$ is a system of parameters in R, then any prime minimal over mS is minimal over $I = (x_1, \dots, x_d)S$:

Suppose P is minimal over mS and $I \subseteq Q \subseteq P$, Q prime.

Then for $\varphi: R \to S$, we have $(x_1, \dots, x_d) \subseteq \varphi^{-1}(I) \subseteq \varphi^{-1}(Q) \subseteq m$, so $\varphi^{-1}(Q) = m \Rightarrow mS \subseteq Q$ so $P = Q$.

The inequality follows from the PIT. $\square$

In fact, we can extend this to a result about the dimension of local R-algebras:

Theorem: Let $\varphi: (R,m) \to (S,n)$ be a map of local rings s.t. $\varphi(m) \subseteq n$. Then

$$\dim S \leq \dim R + \dim S/mS.$$

Pf: Set $d = \dim R$, $e = \dim S/mS$. Let $x_1, \ldots, x_d \in m$ be a system of parameters for R and $y_1, \ldots, y_e \in n$ w/ images a system of parameters for S/mS.

Then for $\alpha \gg 0$,

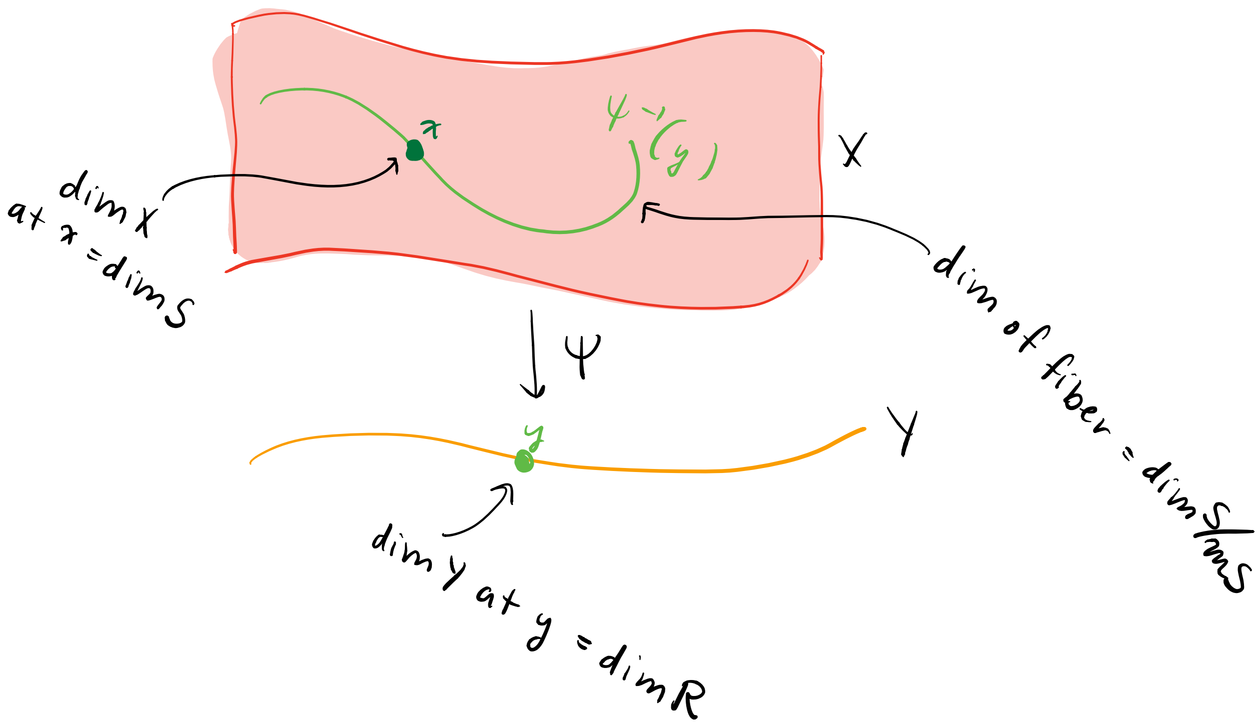
$$n^\alpha \subseteq (y_1, \ldots, y_e) + mS$$

For $\beta \gg 0$, $m^\beta \subseteq (x_1, \ldots, x_d)$, so

$$\begin{aligned} n^{\alpha\beta} &\subseteq ((y_1, \ldots, y_e) + mS)^\beta \\ &\subseteq (y_1, \ldots, y_e) + m^\beta S \\ &\subseteq (x_1, \ldots, x_d, y_1, \ldots, y_e) S, \end{aligned}$$ so

By the PIT, $\dim S \leq d + e$. $\square$

Geometrically, this is saying that if $X \to Y$ is a map of varieties (or schemes), the dimension of X is $\leq$ the dimension of Y + the dimension of a fiber.



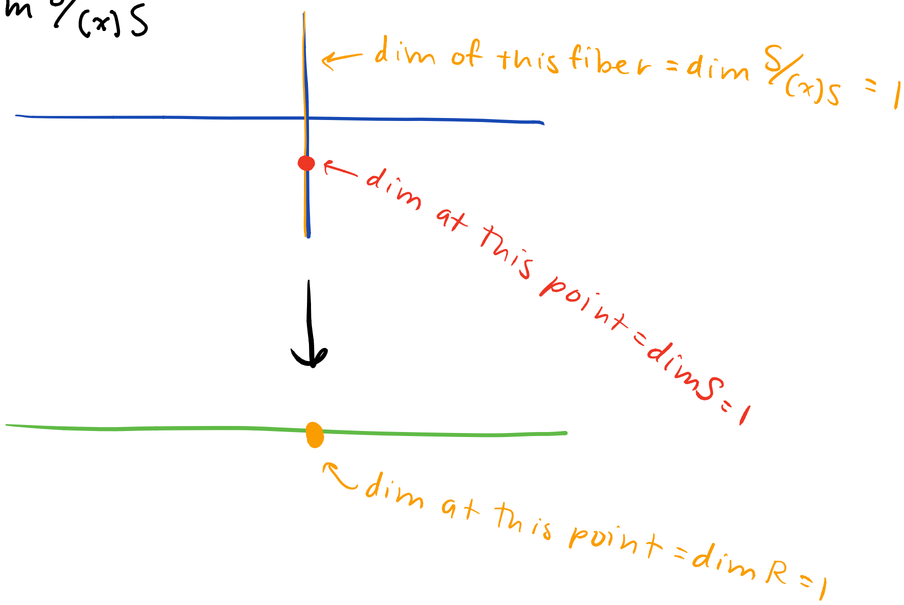
Note that equality doesn't always hold!

Ex: Define $\varphi: \mathbb{C}[x] \to \mathbb{C}[x,y]/(x(y-1))$ and consider the local map

$$\underset{\substack{\| \\ R}}{\mathbb{C}[x]_{(x)}} \longrightarrow \underset{\substack{\| \\ S}}{\mathbb{C}[x,y]_{(x,y)} / (x(y-1))_{(x,y)}} \cong \mathbb{C}[x,y]_{(x,y)}/(x) \cong \mathbb{C}[y]_{(y)}$$

Then $\dim R = \dim S = 1$, and $\dim S/(x)S = \dim S = 1$, so

$\dim S < \dim R + \dim S/(x)S$



In fact, for flat R-algebras (and integral extensions!) <u>equality</u> holds.
To prove this in the flat case, we need the following:

Going Down Theorem (for flat extensions): Let $\varphi: R \to S$ be a map of rings s.t. S is a flat R-module. If $P \supset P'$ are primes of R and Q is a prime of S with $\varphi^{-1}(Q) = P$, then there exists a prime Q' of S contained in Q s.t. $\varphi^{-1}(Q') = P'$.

In fact, Q' may be taken to be any prime of S contained in Q and minimal over $P'S$.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \cup & & \cup \\ P & — & Q \\ \cup & & \cup \\ P' & — & Q' \end{array}$$

Pf: Since $P'S \subseteq Q$, we can find a prime $Q' \subseteq Q$ minimal over $P'S$ (the intersection of a descending chain of primes is prime.

Claim: $S \otimes R/p'$ is flat over R/p'.

Pf of Claim: If $M' \subseteq M$ are R/p' modules, they are R-modules, so $S \otimes_R M' \hookrightarrow S \otimes_R M$.

$$\begin{array}{cc} S \otimes_R M' & S \otimes_R M \\ \| & \| \\ S \otimes_R (R/p' \otimes_{R/p'} M') & S \otimes_R (R/p' \otimes_{R/p'} M) \\ \| & \| \\ (S \otimes_R R/p') \otimes_{R/p'} M' & (S \otimes_R R/p') \otimes_{R/p'} M \end{array} \quad \square$$

Thus $S \otimes R/p' = S/p'S$ is flat over R/p', so we can replace R w/ R/p' and S by $S/p'S$, and reduce to the case $P' = 0$.

S is flat over R, so every nonzerodivisor in R (i.e. every nonzero elt of R since it's an integral domain) is a nonzerodivisor on S (by a cor. we proved to the flatness criterion).

Q' is a minimal prime of S, so it is an associated prime of

$0 \in S$. Thus, $Q' - 0$ consists of zerodivisors on S, so $\varphi^{-1}(Q') = 0$, as desired. $\square$

Note that this implies the standard going down theorem:

If $P_0 \supset P_1 \supset \dots P_n$ is a chain of prime ideals of R

and $R \to S$ s.t. S is a flat R-module, then if Q_0 lies over P_0, we can find a chain

$$Q_0 \supset Q_1 \supset \dots \supset Q_n$$

s.t. Q_i lies over P_i.

Now we can show that equality in the theorem holds for flat algebras:

Cor: Let $\varphi: (R, m) \to (S, n)$ be a map of local rings s.t. the image of m is in n and S is flat as an R-module. Then

$$\dim S = \dim R + \dim S/mS$$

Pf: We already showed one inequality, so we just need to show that $\dim S \geq \dim R + \dim S/mS$.

Let $Q \subseteq S$ be a prime minimal over mS s.t. $\dim Q = \dim S/mS$. Then

$$\dim S \geq \dim Q + \operatorname{codim} Q = \dim S/mS + \operatorname{codim} Q.$$

Thus, it suffices to show $\operatorname{codim} Q \geq \dim R$.

Since Q contains $\mathfrak{m}S$, $\varphi^{-1}(Q) = \mathfrak{m}$. Let $\mathfrak{m} \supsetneq P_1 \supsetneq \cdots \supsetneq P_d$ be a chain of primes in R s.t. $d = \dim R$.

Then, by going down, $\exists$ $Q \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_d$ s.t. $\varphi^{-1}(Q_i) = P_i$.
Thus, $\operatorname{codim} Q \geq \operatorname{codim} \mathfrak{m} = \dim R$. $\square$

We are now finally able to calculate the dimension of a polynomial ring:

Cor: If R is a ring, then $\dim R[x] = 1 + \dim R$. In particular, if k is a field, $\dim k[x_1, \ldots, x_r] = r$.

Pf: The second statement follows from the first by induction on r.

For the first statement, let $P_1 \subset \cdots \subset P_d$ be a chain of primes in R.

Thus, $P_1 R[x] \subset \cdots \subset P_d R[x] \subsetneq \underbrace{P_d R[x] + (x)}$ is a chain of primes in $R[x]$, so $\dim R[x] \geq \dim R + 1$.

$$\left(R[x] \big/ P_d R[x] + (x) \cong R/P_d [x] \big/ (x) \cong R/P_d \right)$$

For the other inequality, it suffices to show that the codim of a maximal ideal in $R[x]$ is $\leq$ the codim of its intersection w/ R $+ 1$. That is, it suffices to prove this more general statement.

Claim: If $P \subseteq R$ is prime, there are primes $Q \subseteq R[x]$ s.t. $Q \cap R = P$, and for a maximal such ideal, we have

$$\dim R[x]_Q = 1 + \dim R_P$$

(Note that if $m \subseteq R[x]$ is maximal, then it will be maximal among primes meeting R in $m \cap R$.)

Pf of claim: let $P \subset R$ prime. Then $PR[x]$ is prime in $R[x]$ and $PR[x] \cap R = P$, which proves the first part of the claim.

Note that if $Q \cap R[x] = P$, then $R - P \subset R[x] - Q$, so $(R_P[x])_Q = R[x]_Q$.

Thus, by replacing R by R_P, we reduce to the case where R is local with maximal ideal P.

Let $Q \subseteq R[x]$ be a maximal ideal containing P. Then $Q \cap R = P$, and we need to show $\operatorname{codim} Q = 1 + \operatorname{codim} P$.

If $P_0 \subset \dots \subset P_d = P$ is a chain of primes in R, then $P_0 R[x] \subset \dots \subset \underset{\substack{\| \\ PR[x]}}{P_d R[x]}$ is a chain of primes of the same length.

$PR[x] \subsetneq PR[x] + (x)$, so $PR[x]$ is not maximal. Thus $Q \supsetneq PR[x]$, so $\operatorname{codim}(Q) \geq d+1$.

For the other inequality, notice that $R \to R[x]_Q$ sends P into Q, so we can apply a previous theorem and get

$$\begin{aligned}\dim R[x]_Q &\le \dim {R[x]_Q}/{PR[x]_Q} + \dim R \\ &= \dim (R/P)[x]_Q + \dim R\end{aligned}$$

But the image of Q in $(R/P)[x]$ is principal since R/P is a field. Thus, its codim is $(R/P)[x]_Q \le 1$, by the PIT, and we're done. $\square$

The claim finishes the proof of the corollary. $\square$